$\Pi_X^{tmp} \supset \Delta_X^{tmp}$

$\Pi_X := (\Pi_X^{tmp})^\wedge$, $\Delta_X := (\Delta_X^{tmp})^\wedge$

$\Delta_X^{tmp} \twoheadrightarrow \underline{\mathbb{Z}}$

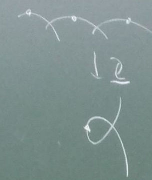
comm. univ. cov.

$\underline{\mathbb{Z}}^\wedge =: \hat{\underline{\mathbb{Z}}}$

$\therefore \Delta_X \twoheadrightarrow \hat{\underline{\mathbb{Z}}}$

$\Delta_X^{\Theta} := \Delta_X / [\Delta_X, [\Delta_X, \Delta_X]]$ theta quotient

ϑ $[\ ,\ ]$: comm. rel



Q

θ

θ

Θ θ

O

§7. Étale Theta Functions — Three Rigidities

§7.1 Theta-Related Varieties

$K/\mathbb{Q}_p$ fin. $\subset \bar{K}$, $G_K := Gal(\bar{K}/K)$

$\mathfrak{X} \to Spf\, O_K$ stable curve of type (1,1) s.t.

special fiber is singular, geom. irred.

the node is rational.

Raynaud gen. fiber is smooth. X

marked pt, special fiber

↑

log str.

Π $\rightsquigarrow$ $\log \cdot \Pi$

...igidités

...nbed pt, special fiber

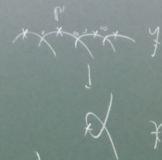
log etc

Π $\log \cdot \Pi$

$$\begin{tikzcd}
\Pi_X^{tmp} \arrow[r, two heads] & (\Pi_X^{tmp})^\Theta \arrow[r, two heads] & (\Pi_X^{tmp})^{ell} \\
\Delta_X^{tmp} \arrow[u, phantom, "\cup"] \arrow[r, two heads] & (\Delta_X^{tmp})^\Theta \arrow[u, phantom, "\cup"] \arrow[r, two heads] & (\Delta_X^{tmp})^{ell} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$$1 \to \Delta_\Theta \to (\Delta_X^{tmp})^\Theta \to (\Delta_X^{tmp})^{ell} \to 1$$

$$1 \to \hat{\mathbb{Z}}(1) \to (\Delta_X^{tmp})^{ell} \to \underline{\mathbb{Z}} \to 1$$



$Y \twoheadrightarrow X$ (resp. $\mathcal{Y} \to \mathcal{X}$) on $\Pi_Y^{tmp} := \ker(\Pi_X^{tmp} \twoheadrightarrow \underline{\mathbb{Z}})$

$\mathrm{Gal}(Y/X) = \underline{\mathbb{Z}}$

$\mathbb{P}^1$ $\mathbb{P}^1$

$$\begin{tikzcd}
\Delta_X^{tmp} \arrow[r, two heads] & (\Delta_X^{tmp})^\Theta \arrow[r, two heads] & \cdots \\
\Delta_Y^{tmp} \arrow[u, phantom, "\cup"] \arrow[r, two heads] & (\Delta_Y^{tmp})^\Theta \arrow[u, phantom, "\cup"] \arrow[r] & \cdots
\end{tikzcd}$$

Θ ϑ θ Θ Θ

$$\Delta_\Theta := \Lambda^2 \Delta_X^{ab} \quad (\cong \hat{\mathbb{Z}}(1))$$

$$\Delta_X^{ell} := \Delta_X^{ab}$$

$$1 \to \Delta_\Theta \to \Delta_X^\Theta \to \Delta_X^{ell} \to 1$$

$$1 \to \hat{\mathbb{Z}}(1) \to \Delta_X^{ell} \to \hat{\mathbb{Z}} \to 1$$

$$\begin{tikzcd}
\Delta_X \arrow[r, two heads] & \Delta_X^\Theta \arrow[r, two heads] & \Delta_X^{ell} \\
\Delta_X^{tmp} \arrow[u, phantom, "\cup"] \arrow[r, two heads] & (\Delta_X^{tmp})^\Theta \arrow[u, phantom, "\cup"] \arrow[r, two heads] & (\Delta_X^{tmp})^{ell} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Δ_X^Θ

theta lies in

anabelian

stable properties of theta

are deduced

from Π_X

$q_X \in \mathcal{O}_K$: q-para...

$N \geq 1$, K_N

$q_X \in \mathcal{O}_K$: q-parameter of X

$N \geq 1$, $K_N := K(\mu_N, q_X^{1/N}) \subset \bar{K}$

arith decomp. gp of a cusp of Y $\rightsquigarrow G_K \to (\Pi_Y^{tp})^{ell}$

section   well-def up to $(\Delta_Y^{tp})^{ell}$-conj.

$$\underbrace{G_{K_N} \hookrightarrow G_K \to (\Pi_Y^{tp})^{ell} \twoheadrightarrow}_{\text{inj} \;/\!/\; \text{def } K_N} (\Pi_Y^{tp})^{ell} / N (\Delta_Y^{tp})^{ell}$$

img is stable under Π_X^{tp}-conj.

$1 \to \mathbb{Z}/N\mathbb{Z}(1) \to (\Delta_X^{tp})^{ell} / N (\Delta_Y^{tp})^{ell} \to \mathbb{Z} \to 1$

triv

G_{K_N}

$$\begin{array}{ccccccccccc}
\Delta_X^{tp} & \twoheadrightarrow & (\Delta_X^{tp})^{\Theta} & \twoheadrightarrow & (\Delta_X^{tp})^{ell} & & \Pi_X^{tp} & \twoheadrightarrow & (\Pi_X^{tp})^{\Theta} & \twoheadrightarrow & (\Pi_X^{tp})^{ell} \\
\cup & & \cup & & \cup & & \cup & & \cup & & \cup \\
\Delta_Y^{tp} & \twoheadrightarrow & \underset{\sim\sim}{(\Delta_Y^{tp})^{\Theta}} & \twoheadrightarrow & \underset{\sim\sim}{(\Delta_Y^{tp})^{ell}} & , & \Pi_Y^{tp} & \twoheadrightarrow & \underline{(\Pi_Y^{tp})^{\Theta}} & \twoheadrightarrow & \underline{(\Pi_Y^{tp})^{ell}}
\end{array}$$

$$1 \to \underset{\substack{/\!/ \\ \text{abelian}}}{\Delta_\Theta} \to (\Delta_Y^{tp})^{\Theta} \to (\Delta_Y^{tp})^{ell} \to 1$$

$(\cong \hat{\mathbb{Z}}(1)^{\otimes 2})$ $(\cong \hat{\mathbb{Z}}(1))$

Δ_X^{Θ}

theta lies in

anabelian properties of theta

Etale theta

are deduced

from Π_X

Y

X $\mathcal{X}$

$\mathbb{P}^1$

$$\Delta_Y^{tmp} \twoheadrightarrow (\Delta_Y^{tmp})^{\Theta} \twoheadrightarrow (\Delta_Y^{tmp})^{ell} \qquad \Pi_Y^{tmp} \twoheadrightarrow (\Pi_Y^{tmp})^{\Theta} \twoheadrightarrow (\Pi_Y^{tmp})^{ell}$$

$$\cup \qquad \cup \qquad \cup \qquad\qquad \cup \qquad \cup \qquad \cup$$

$$\Delta_{Y_N}^{tmp} \twoheadrightarrow \underset{\sim}{(\Delta_{Y_N}^{tmp})^{\Theta}} \twoheadrightarrow \underset{\sim}{(\Delta_{Y_N}^{tmp})^{ell}} \qquad \Pi_{Y_N}^{tmp} \twoheadrightarrow \underset{\sim}{(\Pi_{Y_N}^{tmp})^{\Theta}} \twoheadrightarrow \underset{\sim}{(\Pi_{Y_N}^{tmp})^{ell}}$$

$$1 \to \Delta_\Theta \otimes \mathbb{Z}/N\mathbb{Z} \to (\Pi_{Y_N}^{tmp})^{\Theta}/N(\Delta_Y^{tmp})^{\Theta} \to G_{K_N} \to 1$$

$(\cong \mathbb{Z}/N\mathbb{Z}(1))$

$\mathcal{Y}_N \to \mathcal{Y}$ normalization of $\mathcal{Y}$ in Y_N

$$\rightsquigarrow G_{K_N} \hookrightarrow (\Pi_Y^{tmp})^{ell}/N(\Delta_Y^{tmp})^{ell}$$

ring $\rightsquigarrow Y_N \to Y$

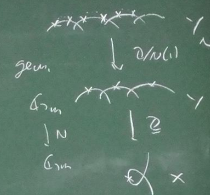
Galois cov

$$1 \to \Pi_{Y_N}^{tmp} \to \Pi_Y^{tmp} \to Gal(Y_N/Y) \to 1$$

$$1 \to (\Delta_Y^{tmp})^{ell} \otimes \mathbb{Z}/N \to Gal(Y_N/Y) \to Gal(K_N/K) \to 1$$

$(\cong \mathbb{Z}/N\mathbb{Z}(1))$

$$\left(\begin{array}{c} \mathbb{Z}/N(1) \\ \cup \\ 1 \to \ker \to Gal(K_N/K) \to (\mathbb{Z}/N)^\times \end{array} \right)$$



$(\Pi_Y^{\mathrm{tp}})^{\Theta} \twoheadrightarrow (\Pi_Y^{\mathrm{tp}})^{\mathrm{ell}}$

$(\Pi_{Y_N}^{\mathrm{tp}})^{\Theta} \twoheadrightarrow (\Pi_{Y_N}^{\mathrm{tp}})^{\mathrm{ell}}$

$\Delta_Y^{\mathrm{tp}})^{\Theta} \to G_{K_N} \to 1$

...tion of $\underline{Y}$ in Y_N

$\mathcal{L}_N \curvearrowright \mathbb{Z} \to \mathbb{Z}$, $a \mapsto 1$

$\Gamma(\underline{Y}_N, \mathcal{O}_{\underline{Y}_N}) = \mathcal{O}_{K_N}$

$J_N := K_N(a^{1/N} \mid a \in K_N) \subset \overline{K}$

$\cup$ fin. Gal.

K_N

$1 \to \Delta_\Theta \otimes \mathbb{Z}/N \to (\Pi_{Y_N}^{\mathrm{tp}})^{\Theta} / N(\Delta_Y^{\mathrm{tp}})^{\Theta} \to G_{K_N} \to 1$

two splittings $\rightsquigarrow H^1(G_{K_N}, \Delta_\Theta \otimes \mathbb{Z}/N)$

def of J_N $\rightsquigarrow$ $H^1(G_{J_N}, \ \ )$

splittings coincide on G_{J_N}

$q_x \in \mathcal{O}_K$

$N \geq 1$,

ang. decomp. gp

Choose an irred. comp. of $\underline{Y}$ as a "basepoint"

$\rightsquigarrow$ irred. comps labelled by $\mathbb{Z}$

deg $\rightsquigarrow$ $\mathrm{Pic}(\underline{Y}_N) \xrightarrow{\sim} \mathbb{Z}^{\mathbb{Z}}$

divisor

$q_x^{1/N}\mathcal{O}_{\underline{Y}_N}$

Cartier divisors on $\underline{Y}_N$

$\mathcal{O}$-Cartier

$\mathrm{Sp} f\, \mathcal{O}_K$

$\underline{Y}_N$

$\downarrow \mathbb{Z}/N(1)$

geom.

$\mathbb{G}_m$ $\underline{Y}$

$\mathbb{Z}$ $\mid \mathbb{Z}$

$\downarrow N$

$\mathbb{G}_m$

α x

$\mathrm{Cl}(K_N/K) \to 1$

$\mathbb{Z}/N(1)$

$1 \to \ker \to \mathrm{Cl}(K_N/K) \to (\mathbb{Z}/N)^{\times}$

$\to 1$

$$\Delta^{+tp}_{Y_N} \to (\Delta^{+tp}_{Y_N})^{\Theta} \to (\Delta^{+tp}_{Y_N})^{ell} \qquad \Pi^{+tp}_{Y_N} \to (\Pi^{+tp}_{Y_N})^{\Theta} \to (\Pi^{+tp}_{Y_N})^{ell}$$

$$\cup \qquad \cup \qquad \cup \qquad\qquad \cup \qquad \cup \qquad \cup$$

$$\Delta^{+tp}_{Z_N} \to \underbrace{(\Delta^{+tp}_{Z_N})^{\Theta}} \to \underbrace{(\Delta^{+tp}_{Z_N})^{ell}} \qquad \Pi^{+tp}_{Z_N} \to (\Pi^{+tp}_{Z_N})^{\Theta} \to (\Pi^{+tp}_{Z_N})^{ell}$$

$\mathfrak{Z}_N \to \mathcal{Y}_N$ normalisation of $\mathcal{Y}$ in Z_N

$s_1 \in \Gamma(\mathcal{Y}, \mathcal{L}_1)$ section zero locus = cusps $(\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = \mathcal{O}_K)$

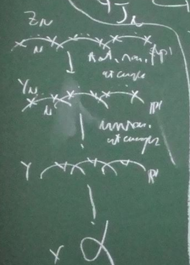
well-def'd up to $\mathcal{O}_K^\times$-multiple

$$\mathcal{L}_N^{\otimes N} \underset{\text{Fix}}{\xrightarrow{\sim}} \mathcal{L}_1|_{\mathcal{Y}_N}$$

$\rightsquigarrow$ identify

$Gal(Y/X) \curvearrowright \mathcal{L}_1 \rightsquigarrow Gal(Y_N/X) \curvearrowright$ unique preserving s_1, $\mathcal{L}_1|_{\mathcal{Y}_N}$

$f \mathcal{O}_K$



$$G_{J_N} \hookrightarrow (\Pi^{+tp}_{Y_N})^{\Theta} / N (\Delta^{+tp}_{Y})^{\Theta}$$

stable under conj. by Π^{+tp}_X

$Z_N \to Y_N$ fin. Gal.

$$1 \to \Pi^{+tp}_{Z_N} \to \Pi^{+tp}_{Y_N} \to Gal(Z_N/Y_N) \to 1$$

$$1 \to \Delta_\Theta \otimes \mathbb{Z}/N \to Gal(Z_N/Y_N) \to Gal(J_N/K_N) \to 1$$

$\ddot{K}_N := K_{2N}$, $\ddot{J}_N := \ddot{K}_N(a^{1/N} \mid a \in \ddot{K}_N) \subset \bar{K}$,

$\ddot{\mathcal{Y}}_N := \mathcal{Y}_{2N}$, $\ddot{Y}_N := Y_{2N} \times_{\ddot{K}_N} \ddot{J}_N$,

$\ddot{\mathcal{L}}_N := \mathcal{L}_N|_{\ddot{\mathcal{Y}}_N} \cong \mathcal{L}_{2N}^{\otimes 2} \times_{O_{\ddot{K}_N}} O_{\ddot{J}_N}$

$\ddot{Z}_N$: the composite of

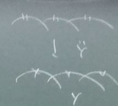
$$\begin{array}{ccc} (\cdot\cdot) & \to & Z_N \\ \downarrow & & \downarrow \\ \ddot{Y}_N & \twoheadrightarrow & Y_N \end{array}$$

$\ddot{\mathfrak{Z}}_N$ normalization of $\mathfrak{Z}_N$ in $\ddot{Z}_N$

$\ddot{Y} := \ddot{Y}_1 = Y_2$, $\ddot{\mathcal{Y}} := \ddot{\mathcal{Y}}_1 = \mathcal{Y}_2$

$\ddot{K} := \ddot{K}_1 = \ddot{J}_1 = K_2$

$(\ddot{Y} := Y_2$



Lem 7.1 ([EtTh, Prop 1.1])

(1). $s_1|_{\mathcal{Y}_N} \in \Gamma(\mathcal{Y}_N, \mathcal{L}_1|_{\mathcal{Y}_N}) = \Gamma(\mathcal{Y}_N, \mathcal{L}_N^{\otimes N})$
has an N-th root $s_N \in \Gamma(\mathfrak{Z}_N, \mathcal{L}_N|_{\mathfrak{Z}_N})$

(2). $\exists$ unique action $\Pi_X^{top} \curvearrowright \mathcal{L}_N \otimes_{O_{K_N}} O_{J_N}$ over $\mathcal{Y}_N \times_{O_{K_N}} O_{J_N}$
which is compat w/ section $s_N : \mathfrak{Z}_N \to \mathcal{L}_N \otimes_{O_{K_N}} O_{J_N}$
factors through $\Pi_X^{top} \twoheadrightarrow \Pi_X^{top}/\Pi_{Z_N}^{top} = \mathrm{Gal}(Z_N/X)$
$\Delta_X^{top}/\Delta_{Z_N}^{top} \curvearrowright \mathcal{L}_N \otimes_{O_{K_N}} O_{J_N}$ faithful

s^{Θ}_{bad} s^{Θ}

mono s^{Θ}

Π_Y^{top}

\mathcal $\mathcal{L}$

\mathfrak $\mathfrak{L}$

$\rightsquigarrow \tau_N : \ddot{\mathcal{Y}}_N \longrightarrow \ddot{\mathcal{L}}_N$

$\uparrow$ zero locus $= \mathfrak{D}_N$ well-def up to $\mathcal{O}^{\times}_{\ddot{\mathcal{Y}}_N}$-multiple

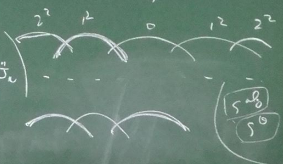
<u>theta trivialisation</u>

$\Pi^{\mathrm{top}}_{\ddot{Y}} \curvearrowright \ddot{\mathcal{Y}}_N, \ddot{\mathcal{L}}_N$

preserves τ_N up to $\mathcal{O}^{\times}_{\ddot{\mathcal{Y}}_N}$-multiple $\left(\rightsquigarrow \Pi^{\mathrm{top}}_{\ddot{Y}} \curvearrowright \ddot{\mathcal{Y}}_N \text{ fixes } \mathfrak{D}_N \right)$

Put $\mathfrak{D}_N := \sum_{j \in \mathbb{Z}} j^2 (\ddot{\mathcal{Y}}_N)_j$

j-th cmp "$q_x^{j^2/2N} = 0$"



<u>Claim</u> $\mathcal{O}_{\ddot{\mathcal{Y}}_N}(\mathfrak{D}_N) \cong \ddot{\mathcal{L}}_N \quad \left(\cong \mathcal{L}^{\otimes 2}_{2N} \otimes_{\mathcal{O}_{\ddot{K}_N}} \mathcal{O}_{\ddot{\mathcal{Y}}_N} \right)$

☺ $\mathrm{Pic}(\ddot{\mathcal{Y}}_N) \cong \mathbb{Z}^{\mathbb{Z}}$

$\rightsquigarrow$ it suffices to show $\underset{\substack{\uparrow \\ \text{inter. prod.}}}{\mathfrak{D}_N \cdot (\ddot{\mathcal{Y}}_N)_i} = 2 \quad \forall i \in \mathbb{Z}$

$0 = \ddot{\mathcal{Y}}_N \cdot (\ddot{\mathcal{Y}}_N)_i = \sum_{j \in \mathbb{Z}} (\ddot{\mathcal{Y}}_N)_j \cdot (\ddot{\mathcal{Y}}_N)_i = 2 + ((\ddot{\mathcal{Y}}_N)_i)^2$

$\Rightarrow (\ddot{\mathcal{Y}}_N)_i^2 = -2$

$\Rightarrow \mathfrak{D}_N \cdot (\ddot{\mathcal{Y}}_N)_i = \sum_{j \in \mathbb{Z}} j^2 (\ddot{\mathcal{Y}}_N)_j \cdot (\ddot{\mathcal{Y}}_N)_i = (i-1)^2 + (i+1)^2 - 2i^2 = 2$ $\lrcorner$

$M \geq 1, \quad M \mid N$

$$\mathcal{Y}_N \twoheadrightarrow \mathcal{Y}_M \twoheadrightarrow \mathcal{Y}$$

$$\ddot{\mathcal{Y}}_N \twoheadrightarrow \ddot{\mathcal{Y}}_M \twoheadrightarrow \ddot{\mathcal{Y}}$$

$$\ddot{\mathfrak{Z}}_N \twoheadrightarrow \mathfrak{Z}_M \twoheadrightarrow \mathcal{Y}$$

$$\mathcal{L}_M|_{\mathcal{Y}_N} \cong \mathcal{L}_N^{\otimes(N/M)}$$

$$\ddot{\mathcal{L}}_M|_{\ddot{\mathcal{Y}}_N} \cong \ddot{\mathcal{L}}_N^{\otimes(N/M)}$$

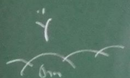
Add $\ddot{J}_N$, $\rightsquigarrow$

$$\begin{tikzcd} \ddot{\mathcal{L}}_N \arrow[r] \arrow[d, "\tau_N"'] \arrow[dr, phantom, "\circlearrowleft"] & \ddot{\mathcal{L}}_M \arrow[d, "\tau_M"] \\ \ddot{\mathcal{Y}}_N \arrow[r] & \ddot{\mathcal{Y}}_M \end{tikzcd}$$

up to $\mathcal{O}^\times_{\ddot{J}_N}$-mult at $\ddot{\mathcal{L}}_N$

$\mathcal{O}^\times_{\ddot{J}_M}$-mult at $\ddot{\mathcal{L}}_M$

$\ddot{\Theta}(-\ddot{U}) = -\ddot{\Theta}(\ddot{U})$ (Lem 7.4 (2), (3))



$\}$ can choose τ_1 so that

$$\Pi^{\mathrm{tp}}_{\ddot{Y}} \curvearrowright \ddot{\mathcal{L}}_1 \quad \text{preserves } \pm\tau_1$$

In summary by def of $\ddot{J}_N$

- By modifying τ_N's by $\mathcal{O}^\times_{\ddot{J}_N}$-multiples $\rightsquigarrow$ may assume $\tau_N^{N/M} = \tau_M$ for $M \mid N$
- In particular, compat. sys. of actions of $\Pi^{\mathrm{tp}}_{\ddot{Y}} \curvearrowright \{\ddot{\mathcal{Y}}_N\}_{N\geq 1}, \{\ddot{\mathcal{L}}_N\}_{N\geq 1}$ which preserves $\{\tau_N\}_{N\geq 1}$
- each action $\Pi^{\mathrm{tp}}_{\ddot{Y}} \curvearrowright \ddot{\mathcal{Y}}_N, \ddot{\mathcal{L}}_N$ differs from the action der'd by $\Pi^{\mathrm{tp}}_{Y} \curvearrowright \mathcal{Y}_N, \mathcal{L}_N \otimes_{\mathcal{O}_{K_N}} \mathcal{O}_{J_N}$ by an elt of $\mu_N(\ddot{J}_N)$ in Lem 7.1 (2)

Rem 7.21 (cf. [E+Th, Prop 1.3])

(1). $\ddot{\eta}^{\Theta}$ arises from a class in

$$
\begin{array}{c}
\varprojlim_N H^1(\Pi^{tp}_{\ddot{Y}}/\Pi^{tp}_{\ddot{Z}_N}, \Delta_\Theta \otimes \mathbb{Z}/N) \\
\downarrow \\
\varprojlim_N H^1(\Delta^{tp}_{\ddot{Y}_N}/\Delta^{tp}_{\ddot{Z}_N}, \Delta_\Theta \otimes \mathbb{Z}/N) \\
\wr\wr \\
\varprojlim_N \mathrm{Hom}(\Delta^{tp}_{\ddot{Y}_N}/\Delta^{tp}_{\ddot{Z}_N}, \Delta_\Theta \otimes \mathbb{Z}/N) \\
\ni \\
\text{natural isom} \\
\{\Delta^{tp}_{\ddot{Y}_N}/\Delta^{tp}_{\ddot{Z}_N} \xrightarrow{\sim} \Delta_\Theta \otimes \mathbb{Z}/N\}_{N \geq 1}
\end{array}
$$

$\ddot{\eta}^{\Theta} \mapsto$

Def 7.2 Take τ_N's as above.

Taking the difference of the conj. cls. sys. of the action of $\Pi^{tp}_{\ddot{Y}} \curvearrowright \{\ddot{Y}_N\}_{N\geq 1}$, $\{\ddot{Z}_N\}_{N\geq 1}$

in Lem 7.1 dot'd by $\{s_N\}_{N\geq 1}$

and the conj. cls. sys. of the action $\Pi^{tp}_{\ddot{Y}} \curvearrowright \{\ddot{Y}_N\}_{N\geq 1}$, $\{\ddot{Z}_N\}_{N\geq 1}$ dot'd by $\{\tau_N\}_{N\geq 1}$

$\rightsquigarrow$ cohom. class $\ddot{\eta}^{\Theta} \in H^1(\Pi^{tp}_{\ddot{Y}}, \Delta_\Theta)$

via isom $\mu_N(\ddot{J}_N) \xrightarrow{\sim} \Delta_\Theta \otimes \mathbb{Z}/N$

isom. scheme theory

§7.2 Étale Theta Function

We recall the sections assoc. to a tangential base pt (cf. [AbsSect, Def 4.1(iii)] & before Def 4.1])

cusp $y \in \ddot{Y}(L)$ L/K fin.

$D_y \subset \Pi_{\ddot{Y}}$ cusp. decomp. gp of y (well-def up to conj)

$$1 \to I_y \to D_y \to G_L \to 1$$

$(\cong \hat{\mathbb{Z}}(1))$

$\mathrm{Sect}(D_y \to G_L)$: the set of splittings of up to conj. by I_y

$\subset$ torsor $H^1(G_L, \hat{\mathbb{Z}}(1)) \cong (L^\times)^\wedge$

(2) $s_2 : \ddot{\mathfrak{X}} \to \ddot{\mathcal{L}}_1$ well-def. up to $\mathcal{O}^\times_{\ddot{K}}$-multiple.

$s_{2N} : \ddot{\mathfrak{X}}_N \to \ddot{\mathcal{L}}_N$: N-th root of s_2,

$\tau_1 : \ddot{\mathfrak{Y}} \to \ddot{\mathcal{L}}_1$ well-def up to $\mathcal{O}^\times_{\ddot{K}}$-multiple

$\tau_N : \ddot{\mathfrak{Y}}_N \to \ddot{\mathcal{L}}_N$: N-th root of τ_1

$\rightsquigarrow \ddot{\eta}^\Theta \in H^1(\Pi^{tp}_{\ddot{Y}}, \Delta_\Theta)$ well-def. up to an $\mathcal{O}^\times_{\ddot{K}}$-multiple.

$\rightsquigarrow \mathcal{O}^\times_{\ddot{K}} \ddot{\eta}^\Theta \subset H^1(\Pi^{tp}_{\ddot{Y}}, \Delta_\Theta)$

subset, indep. of the choices of s_N's, τ_N's

We call any elt <u>étale theta class</u> via Kummer

Sect, Def 4.1 (iii)]
& before Def 4.1]

$y)$

ltiple of
by I_y

$\mathcal{O}^\times_{\ddot{K}}$-multiple

s_N's, t_N's

Kummer

Def 7.3 We call this canonical reduction of the $(L^\times)^\wedge$-torsor $\mathrm{Sect}(D_y \to G_L)$ to the can. $\mathcal{O}_L^\times$-torsor <u>the canonical integral str. of D_y</u>

$s \in \mathrm{Sect}(D_y \to G_L)$ <u>cusp. w/ $\hat{\jmath}$</u>

$\Leftrightarrow$ s comes from a root. of the can. $\mathcal{O}_L^\times$-torsor

$L^\times$-torsor obtained by the push-out of the can. $\mathcal{O}_L^\times$-torsor via $\mathcal{O}_L^\times \to L^\times$ <u>the canonical discrete str. of D_y</u>

$\hat{\mathbb{Z}}' := \hat{\mathbb{Z}}^{(p')}$ $\quad (\mathcal{O}_L^\times)' := \mathrm{Im}(\mathcal{O}_L^\times \to (L^\times)^\wedge \otimes \hat{\mathbb{Z}}')$

$(\mathcal{O}_L^\times)'$-torsor : push-out : via $\mathcal{O}_L^\times \to (\mathcal{O}_L^\times)'$ <u>the canonical tame integral str. of D_y</u>

(cf. [AbsSect, Def 4.1 (ii), (iii)])

ω_y : cotang. of $\ddot{Y}$ at y

$0 \neq \theta \in \omega_y$

take a sys. of N-th roots of any local coord. $t \in \mathfrak{m}_{\ddot{Y},y}$ w/ $dt|_y = \theta$

$\rightsquigarrow$ $\hat{\mathbb{Z}}(1)$ $(\cong I_y)$-torsor $\left(\ddot{Y}|^\wedge_y(t^{1/N})\right)_{N \geq 1} \longrightarrow \ddot{Y}|^\wedge_y$

completion at y

$D_y \to I_y$ $\rightsquigarrow$ $\in \mathrm{Sect}(D_y \to G_L)$

$\rightsquigarrow$ $\mathrm{Sect}(D_y \to G_L)$

$(L^\times)^\wedge$-torsor

canonical

$L^\times$-torsor of non-zero elts $\in \omega_y$

<u>the basis of sect'n assoc. to</u>
<u>the tame base pt θ</u>

$\ddot{Y} \hookleftarrow \ddot{\mathcal{Y}}$ $\quad \mathcal{O}_L$-submodule $\hat{\omega}_y \subset \omega_y$ $\quad \rightsquigarrow$ lin. of $(L^\times)^\wedge$

reduc. $\quad \mathcal{O}_L^\times$ $\quad$ generators of $\hat{\omega}_y$

torsors of

$\ddot{\mathcal{Y}} \supset \mathcal{U} \ (\cong \hat{\mathbb{G}}_m)$

$\uparrow$ $\qquad$ U coord.

$\ddot{U}^2 = U$ $\qquad$ irred. comp. labelled $\in \mathbb{Z}$

\ (nodes)

$$\begin{array}{ccc} \ddot{\mathcal{U}} & \to & \mathcal{U} \\ \cap & \ulcorner & \cap \\ \ddot{\mathcal{Y}} & \to & \mathcal{Y} \end{array}$$

$\ddot{U} \in \Gamma(\ddot{\mathcal{U}}, \mathcal{O}^{\times}_{\ddot{\mathcal{U}}})$

; classical Theta

$\underline{\text{Lem } 7.4}$ ([EtTh, Prop 1.4])

$$\ddot{\Theta}(\ddot{U}) := q_X^{-\frac{1}{8}} \sum_{n \in \mathbb{Z}} (-1)^n q_X^{\frac{1}{2}(n+\frac{1}{2})^2} \ddot{U}^{2n+1} \in \Gamma(\ddot{\mathcal{U}}, \mathcal{O}_{\ddot{\mathcal{U}}})$$

extends uniquely to a mero. fct. on $\ddot{\mathcal{Y}}$.

$$q_X^{-\frac{1}{8}} q_X^{\frac{1}{2}(n+\frac{1}{2})^2} = q_X^{\frac{n(n+1)}{2}} \qquad \frac{n(n+1)}{2} \in \mathbb{Z}$$

(cf. $/\mathbb{C}$ $\quad q := e^{2\pi i \tau}$, $\ddot{U} = e^{\pi i z}$

$$\vartheta_{1,1}(\tau, z) := \sum_{n \in \mathbb{Z}} e^{\pi i \tau (n+\frac{1}{2})^2 + 2\pi i (z+\frac{1}{2})(n+\frac{1}{2})} = \frac{1}{i} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} \ddot{U}^{2n+1}$$
)

We also [illegible] a reduction

$(L^{\times})^n$-torsor $\mathrm{Sect}(D_{\mathcal{Y}} \to G_L)$ to $\{\pm 1\}$-torsor (resp. μ_{2l}-torsor)

$\underline{\{\pm 1\}\text{-str. of } D_{\mathcal{Y}}}$ (resp. $\underline{\mu_{2l}\text{-str. of } D_{\mathcal{Y}}}$)

$s \in \mathrm{Sect}(D_{\mathcal{Y}} \to G_L)$ compat w/ $\quad$ if s comes from a section of the $\{\pm 1\}$-torsor (resp. μ_{2l}-torsor)

$q_X^{-\frac{1}{8}} \, q_X^{\frac{1}{2}(n+\frac{1}{2})^2} = q_X^{\frac{n(n+1)}{2}}$

$^{n+1} \in \Gamma(\ddot{\mathcal{U}}, \mathcal{O}_{\ddot{\mathcal{U}}})$

t. on $\ddot{\mathcal{Y}}$

$)(n+\frac{1}{2}) = \frac{1}{2}\sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} \ddot{\mathcal{U}}^{2n+1}$

(1). $\ddot{\Theta}(\ddot{\mathcal{U}})$ has zeroes of order $=1$ at cusps of $\ddot{\mathcal{Y}}$ & no other zeroes

poles of order $= j^2$ on the irred. comp. labelled j & no other poles

i.e. $\mathrm{div}(\ddot{\Theta}|_\infty = S)$,

(2). $a \in \mathbb{Z}$ $\quad \ddot{\Theta}(\ddot{\mathcal{U}}) = -\ddot{\Theta}(\ddot{\mathcal{U}}^{-1})$, $\quad \ddot{\Theta}(-\ddot{\mathcal{U}}) = -\ddot{\Theta}(\ddot{\mathcal{U}})$

$$\ddot{\Theta}(q_X^{\frac{a}{2}} \ddot{\mathcal{U}}) = (-1)^a q_X^{-\frac{a^2}{2}} \ddot{\Theta}(\ddot{\mathcal{U}})$$

(3). The classes $\mathcal{O}_{\ddot{K}}^{\times} \ddot{\eta}^{\Theta}$ are precisely the Kummer classes assoc. to an $\mathcal{O}_{\ddot{K}}^{\times}$-mult. of the reg. fct $\ddot{\Theta}(\ddot{\mathcal{U}})$ on $\ddot{\mathcal{Y}}$

In particular, for a non-cuspidal pt $y \in \ddot{\mathcal{Y}}(L)$ $\quad L/\ddot{K}$ fin.,

$\mathcal{O}_{\ddot{K}}^{\times} \ddot{\eta}^{\Theta}|_y \in H^1(G_L, \Delta_\Theta) \cong H^1(G_L, \hat{\mathbb{Z}}(1)) \cong (L^\times)^\wedge$

lies in $L^\times \subset (L^\times)^\wedge$ & equal to $\mathcal{O}_{\ddot{K}}^{\times} \ddot{\Theta}(y)$

(4). For a cusp $y \in \ddot{\mathcal{Y}}(L)$ w/ $L/\ddot{K}$ fin.,

$D_y \subset \Pi_{\ddot{\mathcal{Y}}}$ cusp. decomp. at y

Take a sect. $s : G_L \hookrightarrow D_y$ compat. w/ the can. int. str. of D_y

& $\hat{\theta} \in \hat{\omega}_y$ a generator

$\mathcal{O}_{\ddot{K}}^{\times} \ddot{\eta}^{\Theta}|_{s(G_L)} \in H^1(G_L, \Delta_\Theta) \cong H^1(G_L, \hat{\mathbb{Z}}(1)) \cong (L^\times)^\wedge$

(In part. indep. of the choice of the generator $\hat{\theta} \in \hat{\omega}_y$)

lies in $L^\times \subset (L^\times)^\wedge$

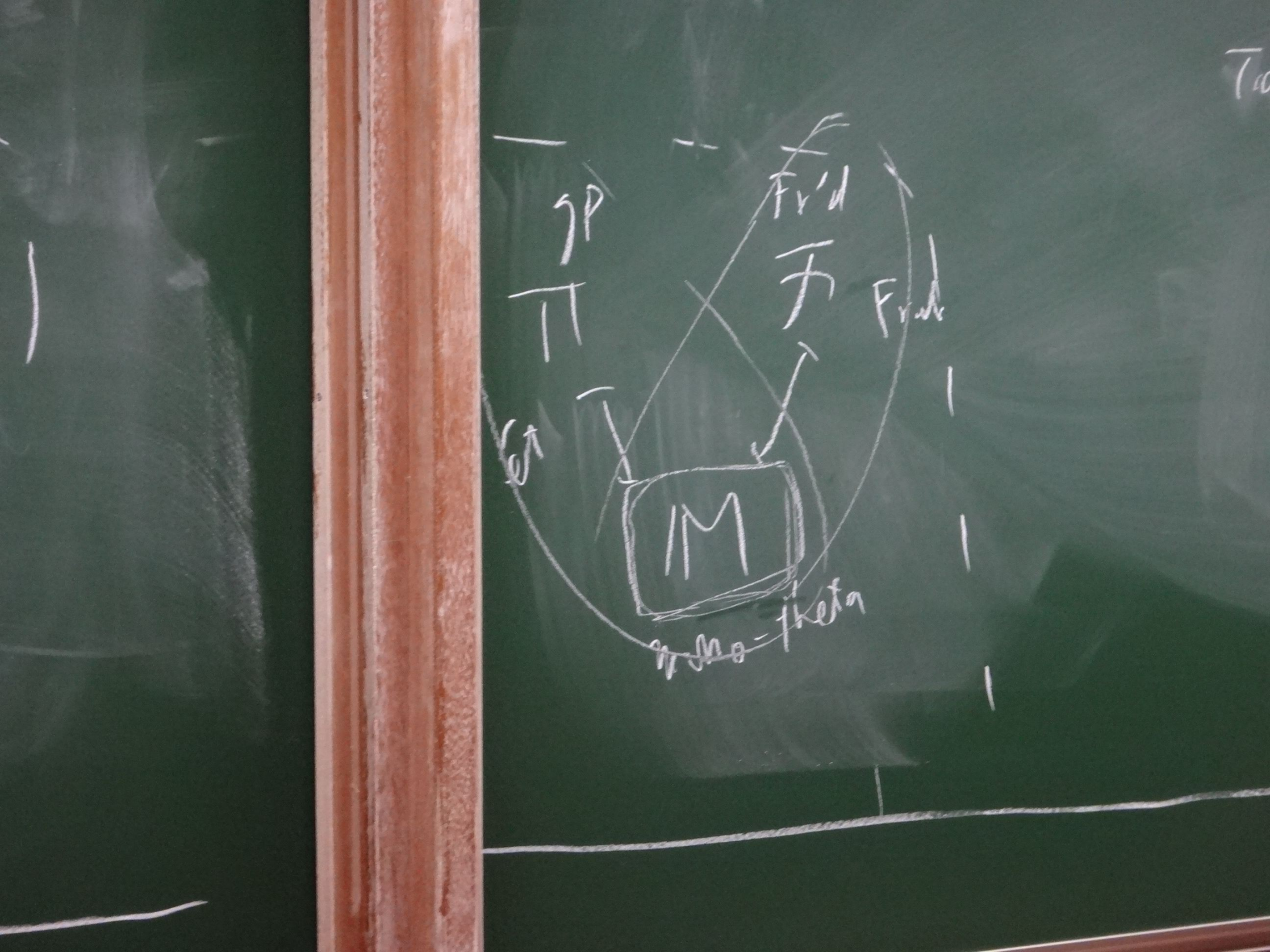
equal to $\mathcal{O}_{\ddot{K}}^{\times} \cdot \frac{d\ddot{\Theta}}{\hat{\theta}}(y)$ — the value at y of the first der. of $\ddot{\Theta}$ at y by $\hat{\theta}$

mp. μ_{2l}-torsor)

from a rotation $\{\pm 1\}$-torsor (resp. μ_{2l}-torsor)

Def 2.3

w_y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Lem 9.5 ([EtTh, Prop 1.5])

$\Delta_\Theta \subset (\Delta^{tp}_{\ddot{Y}})^\Theta \subset (\Pi^{tp}_{\ddot{Y}})^\Theta$

(1). Leray

$$E_2^{a,b} = H^a((\Delta^{tp}_{\ddot{Y}})^{ell}, H^b(\Delta_\Theta, \Delta_\Theta)) \Rightarrow H^{a+b}((\Delta^{tp}_{\ddot{Y}})^\Theta, \Delta_\Theta)$$

$$E_2^{a,b} = H^a(G_{\ddot{K}}, H^b((\Delta^{tp}_{\ddot{Y}})^\Theta, \Delta_\Theta)) \Rightarrow H^{a+b}((\Pi^{tp}_{\ddot{Y}})^\Theta, \Delta_\Theta)$$

deg at E_2 $\rightsquigarrow$

$$H^1((\Pi^{tp}_{\ddot{Y}})^\Theta, \Delta_\Theta) = \mathrm{Fil}^0 \supset \mathrm{Fil}^1 \supset \mathrm{Fil}^2 \supset 0$$

$$\mathrm{Fil}^0/\mathrm{Fil}^1 \cong H_{om}(\Delta_\Theta, \Delta_\Theta) = \hat{\mathbb{Z}}$$

$$\mathrm{Fil}^1/\mathrm{Fil}^2 = H_{om}((\Delta^{tp}_{\ddot{Y}})^\Theta/\Delta_\Theta, \Delta_\Theta) = \hat{\mathbb{Z}} \cdot \log(\ddot{U})$$

$$\mathrm{Fil}^2 \cong H^1(G_{\ddot{K}}, \Delta_\Theta) = H^1(G_{\ddot{K}}, \hat{\mathbb{Z}}(1)) \cong (\ddot{K}^\times)^\wedge$$

id

We call the classes in $\mathcal{O}_{\ddot{K}}^\times \cdot \ddot{\eta}^\Theta$

<u>étale theta function</u> in light of the above relationship of the values of theta fct & the restr. of these classes to G_L via pts

an autom ι of Π_Y^{tp} lying over the action "-1" underlying ell curve of X fixes the irred comp. of $\mathfrak{Y}$ labelled 0

<u>inversion autom</u> of Π_Y^{tp}

(2). $\ddot{\eta}^\Theta$ ...

theta c...

$\mathcal{O}_{\ddot{K}}^\vee \ddot{\eta}$ ...

consider additi...

$\subset (\Pi^{tp}_{\underline{v}})^{\Theta}$

$H^{a+l}((\Delta^{tp}_{\underline{v}})^{\Theta}, \Delta_{\Theta})$

$H^{a+l}((\Pi^{tp}_{\underline{v}})^{\Theta}, \Delta_{\Theta})$

$\supset \mathrm{Fil}^2 \supset 0$

$\mathrm{Hom}((\Delta^{tp}_{\underline{v}})^{\Theta}/\Delta_{\Theta}, \Delta_{\Theta})$

$\hat{\mathbb{Z}} \log(\ddot{U}) \ni \mathrm{id}$

$\cong H^1(G_{\ddot{K}}, \hat{\mathbb{Z}}(1)) \cong (\ddot{K}^{\times})^{\wedge}$

... above

... of theta fct

... of these classes

A_∞ G_L via pts

... ell. curve of X

... labelled 0

▷ inversion autom. of $\Pi^{tp}_{\underline{Y}}$

$$\iota(\ddot{\eta}^{\Theta} + \log(\mathcal{O}^{\times}_{\ddot{K}})) = \ddot{\eta}^{\Theta} + \log(\mathcal{O}^{\times}_{\ddot{K}})$$

$$\iota(\log(\ddot{U}) + \log(\mathcal{O}^{\times}_{\ddot{K}})) = -\log(\ddot{U}) + \log(\mathcal{O}^{\times}_{\ddot{K}})$$

☺ $H^1((\Pi^{tp}_{\underline{v}})^{\Theta}, \Delta_{\Theta}) \to H^1((\Delta^{tp}_{\underline{v}})^{\Theta}, \Delta_{\Theta})^{G_{\ddot{K}}}$

$[\ddot{\Theta}(\ddot{U})] \quad \subset H^1((\Delta^{tp}_{\underline{v}})^{\Theta}, \Delta_{\Theta})$

$\to H^1(\Delta_{\Theta}, \Delta_{\Theta}) = \hat{\mathbb{Z}} \ni 1$

(2) ⇐ fct eq.

(2). ▷ $\ddot{\eta}^{\Theta} \in H^1(\Pi^{tp}_{\underline{v}}, \Delta_{\Theta}) \leftarrow H^1((\Pi^{tp}_{\underline{v}})^{\Theta}, \Delta_{\Theta})$

theta class $\quad \ni \ddot{\eta}^{\Theta} \quad$ $\mathrm{Fil}^0/\mathrm{Fil}^1 = \mathrm{Hom}(\Delta_{\Theta}, \Delta_{\Theta}) \ni \mathrm{id}$

$\mathcal{O}^{\times}_{\ddot{K}} \cdot \ddot{\eta}^{\Theta} \subset H^1((\Pi^{tp}_{\underline{v}})^{\Theta}, \Delta_{\Theta})$

consider additively ~ $\ddot{\eta}^{\Theta} + \log(\mathcal{O}^{\times}_{\ddot{K}})$

$a \in \mathbb{Z} \cong \mathbb{Z} = \Pi^{tp}_{X}/\Pi^{tp}_{\underline{Y}} \curvearrowright \ddot{\eta}^{\Theta} + \log(\mathcal{O}^{\times}_{\ddot{K}})$

$$\ddot{\eta}^{\Theta} - 2a\log(\ddot{U}) - \frac{a^2}{2}\log(q_X) + \log(\mathcal{O}^{\times}_{\ddot{K}})$$

(2). $\gamma \rightsquigarrow$ induces $\Delta_\Theta \xrightarrow{\sim} {}^\dagger\Delta_\Theta$ compatible w/

$$H^1(G_{\ddot{K}}, \Delta_\Theta) \simeq H^1(G_{\ddot{K}}, \hat{\mathbb{Z}}(1)) \xrightarrow{\sim} (\ddot{K}^\times)^\wedge \twoheadrightarrow \hat{\mathbb{Z}}$$

$$H^1(G_{{}^\dagger\ddot{K}}, {}^\dagger\Delta_\Theta) \simeq H^1(G_{{}^\dagger\ddot{K}}, \hat{\mathbb{Z}}(1)) \simeq ({}^\dagger\ddot{K}^\times)^\wedge \twoheadrightarrow \hat{\mathbb{Z}}$$

(3). $\gamma^* : H^1(\Pi^{\text{tmp}}_{\ddot{Y}}, \Delta_\Theta) \simeq H^1(\Pi^{\text{tmp}}_{{}^\dagger\ddot{Y}}, {}^\dagger\Delta_\Theta)$

$$\mathcal{O}^\times_{\ddot{K}}\,\ddot{\eta}^\Theta \xrightarrow[\text{sends}]{} \text{some } {}^\dagger\underline{\mathbb{Z}} \simeq \Pi^{\text{tmp}}_{{}^\dagger X}/\Pi^{\text{tmp}}_{{}^\dagger Y}\text{-conj. of } \mathcal{O}^\times_{{}^\dagger\ddot{K}}\,{}^\dagger\ddot{\eta}^\Theta$$

Prop 7.6 (Anabelian Rigidity of the Étale Theta Function [EtTh, Th 1.6])

X (resp. ${}^\dagger X$) : smooth log-curve of type $(1,1)$ / $K \overset{\text{fin}}{\supset} \mathbb{Q}_p$ (resp. ${}^\dagger K \supset_{\text{fin}}$)

s.t. it has stable red. over $\mathcal{O}_K$ (resp. $\mathcal{O}_{{}^\dagger K}$)

& special fiber is split, geom. irred., node rational.

Similar obj's for ${}^\dagger X$, we use notation ${}^\dagger(\ )$

$$\gamma : \Pi^{\text{tmp}}_X \xrightarrow{\sim} \Pi^{\text{tmp}}_{{}^\dagger X} \quad \text{isom. of top. gps}$$

(1). $\gamma(\Pi^{\text{tmp}}_{\ddot{Y}}) = \Pi^{\text{tmp}}_{{}^\dagger\ddot{Y}}$

(2). $\gamma(\Delta_X^{\mathrm{tp}}) = \Delta_{\dagger X}^{\mathrm{tp}}$

$\rightsquigarrow \gamma(\Delta_\Theta) = {}^\dagger\Delta_\Theta$ wrt of (2) $\Leftarrow$ Cor 6.7, Prop 2.1 (5), (6)

$\uparrow$ Δ_Θ typ thic

(3) After taking some $\Pi_X^{\mathrm{tp}}/\Pi_Y^{\mathrm{tp}} \cong \mathbb{Z}$-conj., we may assume that $\gamma: \Pi_{\ddot{Y}}^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{\dagger\ddot{Y}}^{\mathrm{tp}}$ is compat. w/ suitable inversion autom.

$\rightsquigarrow \gamma: \mathrm{Hom}(\Delta_\Theta, \Delta_\Theta) \longrightarrow \mathrm{Hom}({}^\dagger\Delta_\Theta, {}^\dagger\Delta_\Theta)$ (cf. [SemiAnbd, Th 6.8 (iii)], [AbsTopI, Th 2.3]) (cf. (GC))

$\mathrm{Hom}(\Delta_\Theta, \Delta_\Theta) = F \cdot \mathcal{O}^\times / F \cdot \mathcal{O}'$, $\quad \mathrm{Hom}({}^\dagger\Delta_\Theta, {}^\dagger\Delta_\Theta) = F \cdot \mathcal{O}^\times / F \cdot \mathcal{O}'$

$\mathrm{id} \longmapsto \mathrm{id}$

fixed by ι (resp. $\dagger\iota$) up to $\mathcal{O}_{\ddot{Y}}^\times$-mult. (resp. $\mathcal{O}_{\dagger\ddot{Y}}^\times$-mult) by Cor 7.5 (2)

cal. equiv. bet. Loc's

proof) $\gamma(\Delta_X^{\mathrm{tp}}) = \Delta_{\dagger X}^{\mathrm{tp}}$ by Lem 6.2

$\gamma(\Delta_Y^{\mathrm{tp}}) = \Delta_{\dagger Y}^{\mathrm{tp}}$ ~ discreteness of $\mathbb{Z}$, $\pm\mathbb{Z}$

γ: cusp. decomp. $\to$ cusp decomp. Cor 6.7

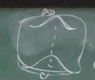
$\ddot{Y} \to Y$ doubling

$\hat{C}_{\mathrm{sum}} \xrightarrow{(2)} \hat{C}_m$

$\downarrow$ $\Leftarrow$ cusp.

$\gamma(\Pi_{\ddot{Y}}^{\mathrm{tp}}) = \Pi_{\dagger\ddot{Y}}^{\mathrm{tp}}$

$\theta(t)$
$\|$
$t^{-\frac{1}{2}}\theta(\frac{1}{t})$

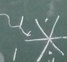


cf. Jacobi's identity

[IUTchI], upper half plane
[IUTchII / HA end.], functional eq.
[IUTchIII / log-shell], analy. cont.

\divideontimes

$\times \div$



$\frac{\ell-1}{2} =: \ell^*$

$\frac{\ell \pm 1}{2} =: \ell^\pm$
$\|$
$|\mathbb{F}_\ell|$

$\mathbb{F}_\ell^* := \mathbb{F}_\ell^\times / \pm 1$

$\mathbb{F}_\ell^{\times\pm}$, $\mathbb{F}_\ell^*$

From now on, we assume

(1). $\ddot{K} = K$.

(2). X w/ marked pt admits a K-core $X \twoheadrightarrow C := X /\!/ \{\pm 1\}$

quotient theory of stacks

(3). $\sqrt{-1} \in K$

$\ddot{X} \twoheadrightarrow X$ Gal cov. deg $= 4$ def'd by mult.-by-2 $(\mathbb{G}_m^{rig}/q_X^2 \xrightarrow{2} \mathbb{G}_m^{rig}/q_X^2)$

$\ddot{\mathfrak{X}} \to \mathfrak{X}$ model

$\ddot{X} \twoheadrightarrow C$ Galois w/ $\mathrm{Gal}(\ddot{X}/C) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$

$\rightsquigarrow \ddot{\eta}^{\Theta}$ (resp. ${}^{\iota}\ddot{\eta}^{\Theta}$) up to a $(\ddot{K}^{\times})^{\wedge}$-mult. (resp. $(\ddot{K}^{\times})^{\wedge}$-mult.)

Actually

It suffices to reduce this $(\ddot{K}^{\times})^{\wedge}$-indet. (resp. $(\ddot{K}^{\times})^{\wedge}$-indet.) to

$\mathcal{O}_{\ddot{K}}^{\times}$-indet. (resp. $\mathcal{O}_{\ddot{K}}^{\times\mu}$-indet.)

by cont. arbitrariness we can show stronger results $\leftarrow$ [EtTh, Rem 1.6.1]

(2). there are two values in $K^\times$ of max. valuations of some std set of values of $\ddot{\eta}^{\underline{\mathbb{Z}} \times \mu_2}$

$$\left(\begin{array}{l} \text{Note } \ddot{\omega}(q_x^{\frac{a}{2}}\sqrt{-1}) = q_x^{-\frac{a^2}{2}} \ddot{\omega}(\sqrt{-1}) \text{ by Lem 7.4 (2)} \\ \ddot{\omega}(-q_x^{\frac{a}{2}}\sqrt{-1}) = -\ddot{\omega}(q_x^{\frac{a}{2}}\sqrt{-1}) \end{array} \right)$$

If they are equal to ± 1, $\Rightarrow$ $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}$: of standard type

---

Choose $\sqrt{-1}, \sqrt{q_x} \in \overline{K}$

(i) $\tau \rightsquigarrow \sqrt{-1} \in K$ $\quad \ddot{\eta}^\Theta$

$\ddot{4}$-tors. μ

$\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2} \subset H^1(\Pi_{\ddot{Y}}^{tp}, \Delta_\Theta)$

$\underline{\mathbb{Z}} \times \mu_2$ - orbit of $\ddot{\eta}^\Theta$

$\Pi_X^{tp} / \Pi_{\ddot{Y}}^{tp}$

Def 7.7 (cf. [EtTh, Def 1.9])

(1). $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}\big|_{\tau}$, $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}\big|_{\tau^{-1}} \subset K^\times$

standard set of values of $\ddot{\eta}^{\Theta, \underline{\mathbb{Z}} \times \mu_2}$

Prop 7.9 (Constant Multiple Rigidity of the Étale Theta Function) [EtTh, Th 1.10]

let $C = X /\!\!/ \{\pm 1\}$ (resp. $^\dagger C = {}^\dagger X /\!\!/ \{\pm 1\}$)

smooth log-orbicurve $/ K$ (resp. $^\dagger K$) $\supset \mathbb{Q}_p$ -fin.

$\ni \sqrt{-1}$ $\quad \ni \sqrt{-1}$

$^\dagger C \leadsto \circledast : {}^\dagger(-)$

$\gamma : \Pi_C^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{^\dagger C}^{\mathrm{tp}}$ isom. of top. gps

(by Lem 7.8 $\leadsto \gamma : \Pi_Y^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{^\dagger Y}^{\mathrm{tp}}$)

Assume γ maps the subset $\ddot{\eta}^{\Theta, \mathbb{Z} \times \mu_2} \subset H^1(\Pi_{\ddot{Y}}^{\mathrm{tp}}, \Delta_\Theta)$

to $^\dagger\ddot{\eta}^{\Theta, \mathbb{Z} \times \mu_2} \subset H^1(\Pi_{^\dagger\ddot{Y}}^{\mathrm{tp}}, {}^\dagger\Delta_\Theta)$

Lem 7.8 ([EtTh, Prop 1.8])

$C = X /\!\!/ \{\pm 1\}$ ($^\dagger C = {}^\dagger X /\!\!/ \{\pm 1\}$) over K (resp. $^\dagger K$) $/ \mathbb{Q}_p$ fin.

resp. smooth log-orb. $\sqrt{-1} \in K$ (resp. $^\dagger K$)

$^\dagger(-)$

$\gamma : \Pi_C^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{^\dagger C}^{\mathrm{tp}} \underset{\text{induces}}{\Longrightarrow} \Pi_X^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{^\dagger X}^{\mathrm{tp}}$, $\Pi_{\ddot{X}}^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{^\dagger\ddot{X}}^{\mathrm{tp}}$

$\Pi_{\ddot{Y}}^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{^\dagger\ddot{Y}}^{\mathrm{tp}}$

(1). γ preserves the property that $\ddot{\eta}^{\Theta, \mathbb{Z} \times \mu_2}$ is of std type.
$\rightsquigarrow$ uniquely det's this collection of classes

(2). $\gamma \rightsquigarrow K^\times \xrightarrow{\sim} {}^\dagger K^\times \subset ({}^\dagger K^\times)^\wedge \cong H^1(G_{{}^\dagger K}, {}^\dagger \Delta_\Theta) \subset H^1(\Pi^{\text{tp}}_{{}^\dagger C}, {}^\dagger \Delta_\Theta)$

$(K^\times)^\wedge \cong H^1(G_K, \Delta_\Theta) \subset H^1(\Pi^{\text{tp}}_C, \Delta_\Theta)$

γ maps the std set of values of $\ddot{\eta}^{\Theta, \mathbb{Z} \times \mu_2}$ to :: of ${}^\dagger \ddot{\eta}^{\Theta, \mathbb{Z} \times \mu_2}$

(3). $\ddot{\eta}^{\Theta, \mathbb{Z} \times \mu_2}$ (hence ${}^\dagger \ddot{\eta}^{\Theta, \mathbb{Z} \times \mu_2}$ as well by (1)) is of std type. K (hence ${}^\dagger K$) res. char > 2

$\Rightarrow \ddot{\eta}^{\Theta, \mathbb{Z} \times \mu_2}$ (resp. ${}^\dagger(\cdot)$) determines a $\{\pm 1\}$-str. on $(K^\times)^\wedge$-torsor (resp. ${}^\dagger(\cdot)$) at the unique cusp of C (resp. ${}^\dagger C$) which is compat. w/ the ram. ind. str. and it is preserved by γ.

$\because$ (1), (3) $\Leftarrow$ (2)

(2): γ induces an isom of dual graph of $\ddot{Y}$ to :: ${}^\dagger \ddot{Y}$ (by Prop 6.5)

ell. cusp'zation (Th 3.7) $\Rightarrow$ γ: dec. gp of pts of $\ddot{Y}$ lying over τ

$\longmapsto$ :: lying over $\tau^{\pm 1}$

$\rightsquigarrow$ (2) //

$l > 2$ prime

$\overline{\Delta}_X := \Delta_X^{\Theta} /$ the subgp gen. by l-th powers of elts of Δ_X^{Θ}

($l \neq 2$ normal)

$\overline{\Delta}_{\Theta} := \mathrm{Im}(\Delta_{\Theta} \to \overline{\Delta}_X)$ $(\cong \mathbb{Z}/l\mathbb{Z}(1))$

$\overline{\Delta}_X^{ell} := \overline{\Delta}_X / \overline{\Delta}_{\Theta}$, $\overline{\Pi}_X := \Pi_X / \ker(\Delta_X \to \overline{\Delta}_X)$

free $\mathbb{Z}/l\mathbb{Z}$-module of rank $= 2$

$\overline{\Pi}^{ell} := \overline{\Pi}_X / \overline{\Delta}_{\Theta}$

x: unique cusp of X, $I_x \subset D_x$

§7.3 l-th Root of Étale Theta Function

X: smooth log-curve of type $(1,1)$ / a field K of char $= 0$

$K \subset \overline{F}$

Assume X admits K-core

$$1 \to \Delta_X \to \Pi_X \to G_K \to 1$$

$\Delta_X^{ell} := \Delta_X^{ab}$, $\Delta_X^{\Theta} := \Delta_X / [\Delta_X, [\Delta_X, \Delta_X]]$

$\Delta_{\Theta} := \mathrm{Im}(\wedge^2 \Delta_X^{ell} \to \Delta_X^{\Theta})$ $\quad 1 \to \Delta_{\Theta} \to \Delta_X^{\Theta} \to \Delta_X^{ell} \to 1$

$\Pi_X^{\Theta} := \Pi_X / \ker(\Delta_X \to \Delta_X^{\Theta})$

$Q \rightsquigarrow \underline{X} \xrightarrow[Q\ (\cong \mathbb{Z}/\ell)]{} X$ $\quad$ ($D_\chi \xrightarrow[\text{triv}]{} Q \rightsquigarrow \underline{X}$ ↪ cusp K-rat'l) ... !

$\overline{\Pi}_{\underline{X}} \subset \overline{\Pi}_X$, $\overline{\Delta}_{\underline{X}} \subset \overline{\Delta}_X$, $\overline{\Delta}^{\mathrm{ell}}_{\underline{X}} \subset \overline{\Delta}^{\mathrm{ell}}_X$

ι, (resp. $\underline{\iota}$) action of X (resp. $\underline{X}$) $\quad$ origin = cusp

$\rightsquigarrow$ mult. by (-1) on the underlying ell. curve

(resp. origin a choice of cusp)

$C := X /\!/ \iota$, $\underline{C} := \underline{X} /\!/ \underline{\iota}$

$\Delta_C \subset \Pi_C$, $\Delta_{\underline{C}} \subset \Pi_{\underline{C}}$

$$\begin{tikzcd} D_\chi \arrow[r, hook] & \Pi^{\Theta}_X \\ I_\chi \arrow[u, phantom, "\vee" description] \arrow[r, "\sim"] & \Delta_\Theta \arrow[u, phantom, "\cup" description] \end{tikzcd}$$

$\overline{D}_\chi := \mathrm{Im}(D_\chi \to \overline{\Pi}_X)$

$1 \to \underset{(\cong \hat{\mathbb{Z}}(1))}{\overline{\Delta}_\Theta} \to \overline{D}_\chi \to G_{k_1} \to 1$

<u>Assumption (1)</u> Choose a quot. $\overline{\Pi}^{\mathrm{ell}}_X \twoheadrightarrow Q$ ← free $\mathbb{Z}/\ell$-mod of rk = 1

s.t.

$$\begin{tikzcd} \overline{\Pi}^{\mathrm{ell}}_X \arrow[r, two heads] & Q \\ \overline{\Delta}^{\mathrm{ell}}_X \arrow[u, phantom, "\cup" description] \arrow[ur, two heads] & \\ D_\chi \arrow[u, phantom, "\cup" description] \arrow[uur, "\text{triv}"'] & \end{tikzcd}$$

eigenvalue -1 (resp. $+1$)

$\rightsquigarrow \overline{\Delta}^{ell}_{\underline{X}}$ (resp. $\overline{\Delta}_{\Theta}$)

$\rightsquigarrow \overline{\Delta}_{\underline{X}} \cong \overline{\Delta}^{ell}_{\underline{X}} \times \overline{\Delta}_{\Theta}$ (cf. [EtTh, Prop 2.2 (i)])

compat w/ conj. action of $\overline{\Pi}_{\underline{X}}$ ($\leftarrow \iota_{\underline{C}}$ contains $\overline{\Pi}_{\underline{X}}$-conj.)

$s_?: \overline{\Delta}^{ell}_{\underline{X}} \hookrightarrow \overline{\Delta}_{\underline{X}}$ the splitting of $\overline{\Delta}_{\underline{X}} \twoheadrightarrow \overline{\Delta}^{ell}_{\underline{X}}$

$\overline{\Pi}_{x} \cong \overline{\Pi}_{\underline{X}} / \mathrm{Im}(s_?)$ $/G_K$

$\iota \curvearrowright Q$ acts by (-1) $(l \neq 2)$

$\forall$ gen. $\mathrm{Gal}(\underline{X}/X)$ does not descend to an autom. of $\underline{C}$ over C

$\rightsquigarrow \underline{C} \longrightarrow C$ not Galois (cf. [EtTh, Rem 2.1.1])

$\overline{\Pi}_C := \Pi_C / \ker(\Pi_X \twoheadrightarrow \overline{\Pi}_X)$ $\quad \overline{\Delta}_C$

$\overline{\Delta}_C$, $\overline{\Delta}^{ell}_C := \overline{\Delta}_C / \ker(\overline{\Delta}_X \twoheadrightarrow \overline{\Delta}^{ell}_X)$

$\overline{\Pi}_{\underline{C}} := \Pi_{\underline{C}} / \ker(\Pi_{\underline{X}} \twoheadrightarrow \overline{\Pi}_{\underline{X}})$

Assumption (2) Choose $\iota_{\underline{C}} \in \overline{\Delta}_{\underline{C}}$ which lifts the $(1\neq)$ elt $\in \mathrm{Gal}(\underline{X}/\underline{C}) \cong \mathbb{Z}/2\mathbb{Z}$

$\rightsquigarrow \iota_{\underline{C}} \curvearrowright_{\text{conj.}} \overline{\Delta}_{\underline{X}}$ free of rk $= 2$ / $\mathbb{Z}_l$

Assumption(3) Choose any elt $s^{\text{Assump}(3)} \in H^1(G_K, \overline{\Delta}_\Theta) \cong K^\times/(K^\times)^\ell$ - torsor

$\mathrm{Sect}(\overline{D}_x \twoheadrightarrow G_K)$

$\rightsquigarrow \Pi_{\underline{X}} \twoheadrightarrow \overline{\Pi}_{\underline{X}} \twoheadrightarrow \overline{\Pi}_{\underline{X}} / \mathrm{Im}(s_2) \cong \overline{D}_x \twoheadrightarrow \overline{D}_x / s^{\text{Assump}(3)}(G_K) \cong \overline{\Delta}_\Theta$

$\rightsquigarrow \underline{\underline{X}} \to \underline{X}$, $\mathrm{Gal} = \overline{\Delta}_\Theta$ $(\cong \mathbb{Z}/\ell(1))$

$\overline{\Delta}_{\underline{\underline{X}}} \subset \overline{\Delta}_{\underline{X}}$, $\overline{\Pi}_{\underline{\underline{X}}} \subset \overline{\Pi}_{\underline{X}}$

$\overline{\Delta}_{\underline{\underline{X}}} = \mathrm{Im}(s_2)$

$\overline{\Delta}_{\underline{X}} = \overline{\Delta}_{\underline{\underline{X}}} \cdot \overline{\Delta}_\Theta$

$\ell > 2$ prime

$\overline{\Delta}_X := \Delta_X^\Theta /$ the subgp gen. by ℓ-th powers of elts of Δ_X^Θ (char. normal)

$\overline{\Delta}_\Theta := \mathrm{Im}(\Delta_\Theta \to \overline{\Delta}_X)$ $(\cong \mathbb{Z}/\ell(1))$

$\overline{\Delta}_X^{\mathrm{ell}} := \overline{\Delta}_X / \overline{\Delta}_\Theta$, $\overline{\Pi}_X := \Pi_X / \mathrm{Ker}(\Delta_X \to \overline{\Delta}_X)$

free $\mathbb{Z}/\ell$-module of rank $= 2$

$\overline{\Pi}^{\mathrm{ell}} := \overline{\Pi}_X / \overline{\Delta}_\Theta$

x: unique cusp of X, $I_x \subset D_x$

Lem 7.12 ([EtTh, Prop 2.4])

$\underline{\underline{X}}$ (resp. $^{\dagger}\underline{\underline{X}}$) smooth log curve / K (resp. $^{\dagger}K$) $\supseteq_{\text{fin}} \mathbb{Q}_p$

constr'd as above.

$^{\dagger}\underline{\underline{X}} \rightsquigarrow {}^{\dagger}(-)$ X, (resp. $^{\dagger}X$) has stable red over O_K (resp. $O_{^{\dagger}K}$)

sp. fiber singular, geom. irred. w/ node

$\gamma: \Pi^{\text{tp}}_{\underline{\underline{X}}} \xrightarrow{\sim} \Pi^{\text{tp}}_{^{\dagger}\underline{\underline{X}}}$ isom of tp. gps

$\rightsquigarrow$ induces $\Pi^{\text{tp}}_{C} \xrightarrow{\sim} \Pi^{\text{tp}}_{^{\dagger}C}$, $\Pi^{\text{tp}}_{\underline{C}} \xrightarrow{\sim} \Pi^{\text{tp}}_{^{\dagger}\underline{C}}$,

$\Pi^{\text{tp}}_{X} \xrightarrow{\sim} \Pi^{\text{tp}}_{^{\dagger}X}$, $\Pi^{\text{tp}}_{\underline{X}} \xrightarrow{\sim} \Pi^{\text{tp}}_{^{\dagger}\underline{X}}$, $\Pi^{\text{tp}}_{\ddot{Y}} \xrightarrow{\sim} \Pi^{\text{tp}}_{^{\dagger}\ddot{Y}}$

$I_x \cong \Delta_\Theta \twoheadrightarrow \overline{\Delta}_\Theta \rightsquigarrow \underline{\underline{Y}} \twoheadrightarrow \underline{X}$

tot. ram. at the cusps

image of $\iota_{\underline{C}}$ in $\overline{\Delta}_C / \overline{\Delta}_{\underline{X}}$ char'ed as the unique coset of $\overline{\Delta}_C / \overline{\Delta}_{\underline{X}}$

which lifts the $(1\neq)$ elt $\in \overline{\Delta}_C / \overline{\Delta}_X$

normalizes the subgp $\overline{\Delta}_{\underline{\underline{X}}} \subset \overline{\Delta}_C$

(cf. [EtTh, Prop 2.2 (ii)]) $\left(\Delta_{\underline{X}} / \overline{\Delta}_{\underline{\underline{X}}} \cong \overline{\Delta}_\Theta \right.$ eigenvalue $= 1$ $\left. \rightsquigarrow \overline{\Delta}_\Theta \right)$

loc l

X bad $\overset{}{=}$ good X

Def 7.11 ([EtTh, Def 2.5])

Assume K : res char > 2, $\ddot{K} = K$

Make two assumptions

Assumption (4) $\Pi^{\mathrm{ell}}_X \twoheadrightarrow Q$ factors through the nat. quot. $\Pi_X \twoheadrightarrow \hat{\mathbb{Z}}$ det'd by the quot $\Pi_X^{\mathrm{tp}} \twoheadrightarrow \mathbb{Z}$

Assumption (5) $s^{\mathrm{Aug}(3)} \in \mathrm{Sect}(\bar{D}_X \to G_K)$ is compat w/ the $\{\pm 1\}$-action.

cf Prop 7.9(3)

---

local

bad $\underline{X}$ = good $\underline{X}$

global

$\underline{X}_K$, $\underline{C}_K$

$\mathbb{F}_\ell^{\rtimes\pm}$, $\mathbb{F}_\ell^{*}$

$\mathbb{F}_\ell^{\rtimes\pm} = \mathbb{F}_\ell \rtimes \{\pm 1\}$

Rem 7.10.1 ([EtTh, Rem 2.6.1])

$\mu_\ell \subset K$

Lem 7.10 $\rightsquigarrow$

$\mathrm{Aut}_K(\underline{\underline{X}}) = \mu_\ell \times \{\pm 1\}$

$\mathrm{Aut}_K(\underline{X}) = \mathbb{Z}/\ell\mathbb{Z} \rtimes \{\pm 1\}$

$\mathrm{Aut}_K(\underline{C}) = \{1\}$

$1, 2, \dots, \ell^* = \frac{\ell - 1}{2}$

$\{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset \dots \subset \{1, \dots, \ell^*\}$

$$
\begin{tikzcd}
\underline{\underline{\ddot{Y}}} \arrow[d, "\mu_2"] \arrow[rrrd, bend left, "\Delta_\Theta (\cong \mathbb{Z}/\ell(1))"] & & & \\
\underline{\underline{Y}} \arrow[rr] \arrow[d, "\ell\mathbb{Z}"] & & Y \arrow[d, "\mathbb{Z}"] & \ddot{Y} \arrow[l, "\mu_2"'] \arrow[d, "2\mathbb{Z}"] \\
\underline{\underline{X}} \arrow[r, "\Delta_\Theta"', "(\cong \mathbb{Z}/\ell(1))"] & \underline{X} \arrow[r, "Q"', "(\cong \mathbb{Z}/\ell)"] \arrow[d, "\{\pm 1\}"'] & X \arrow[d, "\{\pm 1\}"] & \ddot{X} \arrow[l, "\text{ext'n of } \mathbb{Z}/2 \text{ by } \mu_2"'] \\
 & \underline{C} \arrow[r, "\text{non-Gal}"'] & C &
\end{tikzcd}
$$

A smooth $O_?$-orbicurve $\mathbb{K}$ is called

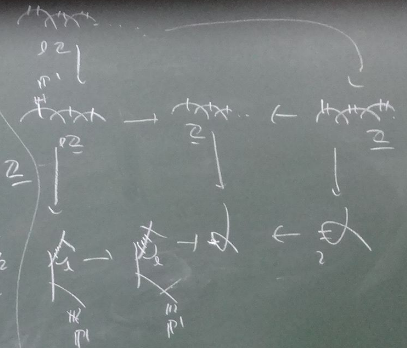
$\underline{\text{of type } (1, \mathbb{Z}/\ell\mathbb{Z})}$ (resp. $\underline{\text{of type } (1, (\mathbb{Z}/\ell)^{\Theta})}$)

if it is isom. to $\underline{X}$ (resp. $\underline{\underline{X}}$)

$\underline{\underline{Y}} \to X$ (resp. $\underline{\underline{\ddot{Y}}} \to X$) the composite of the covering $Y \to X$ (resp. $\ddot{Y} \to X$)

w/ $\underline{\underline{X}} \to X$

$$\begin{tikzcd}
\underline{\underline{Y}} \arrow[d, "\mu_2"] \arrow[rrr, bend left, "\overline{\Delta}_\Theta \cong \mathbb{Z}/\ell(1)"] & & & \\
\underline{Y} \arrow[d, "\ell\mathbb{Z}"] \arrow[rr] & & Y \arrow[d, "\mathbb{Z}"] & \ddot{Y} \arrow[l, "\mu_2"'] \arrow[d, "2\mathbb{Z}"] \\
\underline{\underline{X}} \arrow[r, "\overline{\Delta}_\Theta"', "\cong \mathbb{Z}/\ell(1)"] & \underline{X} \arrow[d, "\{\pm 1\}"] \arrow[r, "Q \cong (\mathbb{Z}/\ell)"] & X \arrow[d, "\{\pm 1\}"] & \ddot{X} \arrow[l, "\text{ext'n of } \mathbb{Z}/2 \text{ by } \mu_2"'] \\
& \underline{C} \arrow[r, "\text{non-Gal}"'] & C &
\end{tikzcd}$$



A smooth $\mathcal{O}_k$-[illegible] $\mathbb{K}$ is called

of type $(1, \mathbb{Z}/\ell\mathbb{Z})$ (resp. of type $(1, (\mathbb{Z}/\ell)^{\Theta})$)

if it is isom. to $\underline{X}$ (resp. $\underline{\underline{X}}$)

$\underline{Y} \to X$ (resp. $\underline{\underline{Y}} \to X$) the composite of the covering $Y \to X$ (resp. $\ddot{Y} \to X$)

w/ $\underline{X} \to X$

By Assmp (4) $\rightsquigarrow$ $\ddot{Y} \to X$, $\ddot{Y} \dashrightarrow \underline{X} \to X$

$\ddot{\eta}^{\Theta} \in H^1(\Pi^{\text{tp}}_{\ddot{Y}}, \Delta_\Theta)$

well-def'd up to a $\mathbb{Q}_\pi^\times$-mult.

$\ddot{\eta}^{\Theta} \in H^1(\Pi^{\text{tp}}_{\ddot{Y}}, \Delta_\Theta \otimes \mathbb{Z}/l)$ (by Lem 7.5 (2))

$\Pi^{\text{tp}}_{\underline{X}} / \Pi^{\text{tp}}_{\ddot{Y}} \cong \underline{\mathbb{Z}} \times \mu_2$ which can be regarded as alg's assoc. to $\Pi^{\text{tp}}_{\underline{X}}$

$\ddot{\eta}^{\Theta} \in H^1((\Pi^{\text{tp}}_{\ddot{Y}})^{\Theta}, \Delta_\Theta \otimes \mathbb{Z}/l)$

$(\Pi^{\text{tp}}_{\ddot{Y}})^{\Theta} \twoheadleftarrow \Pi^{\text{tp}}_{\ddot{Y}} \leftarrow D_X$

$H^1(D_X, \Delta_\Theta \otimes \mathbb{Z}/l)$

$\in H^1(\overline{D}_X, \Delta_\Theta \otimes \mathbb{Z}/l)$

$\ddot{\eta}^{\Theta} \in H^1(\overline{D}_X, \Delta_\Theta \otimes \mathbb{Z}/l)$

special fiber

- X : 1 irred. comp ($\cong \mathbb{P}^1$) 1 cusp on it
- $\ddot{X}$: 2 irred. comp ($\cong \mathbb{P}^1$) 2 cusps on each
- $\underline{X}$: l irred. comp ($\cong \mathbb{P}^1$) 1 cusp on each
- $\underline{X}$: l irred. comp ($\not\cong \mathbb{P}^1$) 1 cusp on each
- $\underline{\underline{X}}$: l irred. comp ($\not\cong \mathbb{P}^1$) 1 cusp on each
- Y : irred. comp ($\cong \mathbb{P}^1$) para'sed by $\underline{\mathbb{Z}}$ 2 cusps on each
- $\ddot{Y}$: irred. comp ($\cong \mathbb{P}^1$) by $\underline{\mathbb{Z}}$ 1 cusp on each
- $\underline{Y}$: irred. comp ($\not\cong \mathbb{P}^1$) by $l\underline{\mathbb{Z}}$ 1 cusp on each
- $\underline{\underline{Y}}$: irred. comp ($\not\cong \mathbb{P}^1$) by $l\underline{\mathbb{Z}}$ 2 cusps on each
- $\underline{\underline{\ddot{Y}}}$

$-\mathrm{mult}.$

$\ldots$ as alg's assoc. to $\Pi_{\underline{X}}^{\mathrm{tp}}$

$\Pi_{\underline{Y}}^{\mathrm{tp}} \leftarrow D_X$

$H^1(G_k, \Delta_\Theta \otimes \mathbb{Z}/l) \leftarrow \ni \Theta$, $(k^\times)/(k^\times)^l$

$0 \to H^1(\bar{D}_X, \Delta_\Theta \otimes \mathbb{Z}/l) \to H^1(D_X, \Delta_\Theta \otimes \mathbb{Z}/l) \to \mathrm{Hom}(l\Delta_\Theta, \Delta_\Theta \otimes \mathbb{Z}/l)$

$\mathrm{Im} \ni s$, $\mathrm{Im}(s) \ni \ddot{\eta}^\Theta$

$\mathrm{Sect}(\bar{D}_X \to G_k)$

$H^1(\Pi_{\underline{Y}}^{\mathrm{tp}})^\Theta, \Delta_\Theta \otimes \mathbb{Z}/l)$

$H^1(\Delta_\Theta \otimes \mathbb{Z}/l, \Delta_\Theta \otimes \mathbb{Z}/l)$
Hom $\ni \mathrm{id}$

id by Lem 1, $s(1)$

$g \mapsto g(s(\bar{g})) = g(s(1))^{-1} = g \cdot 1^{-1} = g$

$\Delta_\Theta \otimes \mathbb{Z}/l$

$\rightsquigarrow$ $\mathrm{Im} \ni \ddot{\eta}^\Theta$ differ from $\ddot{\eta}^\Theta \in H^1(\bar{D}_X, \Delta_\Theta \otimes \mathbb{Z}/l)$ by an $H^1(G_k, \Delta_\Theta \otimes \mathbb{Z}/l) \cong k^\times/(k^\times)^l$ -multiple.

$s^{\mathrm{Assump}(3)} \in \mathrm{Sect}(\bar{D}_X \to G_k)$

☺

$0 \to H^1(\bar{D}_X, \Delta_\Theta \otimes \mathbb{Z}/l) \to H^1(D_X, \Delta_\Theta \otimes \mathbb{Z}/l) \to H^1(l\Delta_\Theta, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$

$\cong \mathrm{Hom}(l\Delta_\Theta, \Delta_\Theta \otimes \mathbb{Z}/l)$

$\Theta \mapsto \ddot{\eta}^\Theta \mapsto 0$

$s \in \mathrm{Sect}(\bar{D}_X \to G_k)$,

$\bar{D}_X \ni g \mapsto g(s(\bar{g}))^{-1}$ 1-cocycle $\rightsquigarrow \Theta \in H^1(\bar{D}_X, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$

$G_k \ni \bar{g}$

$\mathrm{Sect}(\bar{D}_X \to G_k) \xrightarrow[\text{map}]{} H^1(\bar{D}_X, \Delta_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$

$\mathrm{Im}(s^{\mathrm{Assump}(3)}) \in H^1(\Pi_X, \Delta_\Theta \otimes \mathbb{Z}/\ell)$

We can modify $\ddot{\eta}^\Theta \in H^1(\Pi_X, \Delta_\Theta \otimes \mathbb{Z}/\ell)$ by a $K^\times$-multiple

well-def up to $(K^\times)^\ell$-multiple,

coincide w/ $\mathrm{Im}(s^{\mathrm{Assump}(3)}) \in H^1(\Pi_X, \Delta_\Theta \otimes \mathbb{Z}/\ell)$

stronger claim also holds, i.e.

we can modify $\ddot{\eta}^\Theta$ by an $\mathcal{O}_K^\times$-mult.

well-def up to $(\mathcal{O}_K^\times)^\ell$-multiple to make it coincide w/ $\mathrm{Im}(s^{\mathrm{Assump}(3)})$,

since $s^{\mathrm{Assump}(3)}$ is compat w/ the can. int. str. of D_X by Assump (5).

As a concl. $\rightsquigarrow$ by modify $\ddot{\eta}^\Theta \in H^1((\Pi_{\ddot{Y}}^{\mathrm{tp}})^\Theta, \Delta_\Theta \otimes \mathbb{Z}/\ell)$

by a $\mathcal{O}_K^\times$-multiple

which is well-def up to $(\mathcal{O}_K^\times)^\ell$-multiple. ($\leftarrow$ not $\mathcal{O}_K^\times$-mult.)

we can and shall assume that

$\ddot{\eta}^\Theta = \mathrm{Im}(s^{\mathrm{Assump}(3)}) \in H^1(\Pi_X, \Delta_\Theta \otimes \mathbb{Z}/\ell)$

(i.e. by the choice of $\underline{\underline{X}}$ $\quad s_\ell \quad \iota_\ell$)

$\underline{\underline{X}} \twoheadrightarrow \underline{X}$ the covering of "taking ℓ-th root of Θ"

$H_{\mathrm{tp}}(\ell\Delta_\Theta, \Delta_\Theta \otimes \mathbb{Z}/\ell)$

$= \gamma \cdot 1^{-1} = \gamma$

by an

$, \Delta_\Theta \otimes \mathbb{Z}/\ell) \cong K^\times/(K^\times)^\ell$ -multiple.

$\Delta_\Theta \otimes \mathbb{Z}/\ell\mathbb{Z})$

$\ell\Delta_\Theta, \Delta_\Theta \otimes \mathbb{Z}/\ell)$

$\Delta_\Theta \otimes \mathbb{Z}/\ell\mathbb{Z})$

$(\Pi_X, \Delta_\Theta \otimes \mathbb{Z}/\ell\mathbb{Z})$

mono-theta env

cycl. rig.

const. mult. rig.

disc. rig.

$\Pi_X \to G \mapsto g(S($

μ_l

$O^{\times}/\mu$

$\to G$

splitting

$\mathcal{O}^{\times}$

Π_T

M